Section 7.2: The Definition of the Laplace Transform

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What is the Laplace Transform?

Discussion

- **Operator (like the derivative)**
- **Notation/Letters**

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<u>What is the Laplace Transform?</u>

Laplace Transform

Definition 1. Let $f(t)$ be a function on $[0, \infty)$. The **Laplace transform** of f is the function F defined by the integral

$$(1) \qquad F(s) := \int_0^\infty e^{-st} f(t)\, dt\,.$$

The domain of $F(s)$ is all the values of s for which the integral in (1) exists.† The Laplace transform of f is denoted by both F and $\mathscr{L}\{f\}$.

More Discussion

- **Fill in blank space**
- **Treat *s* like a constant…**
- **Domains**

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Calculating the Laplace Transform of Some Functions

Example 1 Determine the Laplace transform of the constant function $f(t) = 1, t \geq 0$.

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Example 2 Determine the Laplace transform of $f(t) = e^{at}$, where a is a constant.

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Example 3 Find $\mathscr{L}\{\sin bt\}$, where b is a nonzero constant.

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Calculating the Laplace Transform of Some Functions

Example 4 Determine the Laplace transform of $f(t) = \begin{cases} 2, & 0 < t < 5, \\ 0, & 5 < t < 10, \\ e^{4t}, & 10 < t. \end{cases}$

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Some Properties of the Laplace Transform: LINEARITY

Linearity of the Transform

Theorem 1. Let f, f_1, and f_2 be functions whose Laplace transforms exist for $s > \alpha$ and let c be a constant. Then, for $s > \alpha$,

(2) $$\mathscr{L}\{f_1 + f_2\} = \mathscr{L}\{f_1\} + \mathscr{L}\{f_2\} ,$$

(3) $$\mathscr{L}\{cf\} = c\mathscr{L}\{f\} .$$

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Some Properties of the Laplace Transform: LINEARITY

Example 5 Determine $\mathcal{L}\{11 + 5e^{4t} - 6\sin 2t\}$.

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Table of Laplace Transforms

TABLE 7.1 Brief Table of Laplace Transforms

$f(t)$	$F(s) = \mathscr{L}\{f\}(s)$
1	$\dfrac{1}{s}, \quad s > 0$
e^{at}	$\dfrac{1}{s-a}, \quad s > a$
$t^n, \quad n = 1, 2, \ldots$	$\dfrac{n!}{s^{n+1}}, \quad s > 0$
$\sin bt$	$\dfrac{b}{s^2+b^2}, \quad s > 0$
$\cos bt$	$\dfrac{s}{s^2+b^2}, \quad s > 0$
$e^{at}t^n, \quad n = 1, 2, \ldots$	$\dfrac{n!}{(s-a)^{n+1}}, \quad s > a$
$e^{at}\sin bt$	$\dfrac{b}{(s-a)^2+b^2}, \quad s > a$
$e^{at}\cos bt$	$\dfrac{s-a}{(s-a)^2+b^2}, \quad s > a$

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Table of Laplace Transforms

Example 6 Use Table 7.1 to determine $\mathscr{L}\{5t^2e^{-3t} - e^{12t}\cos 8t\}$.

TABLE 7.1 Brief Table of Laplace Transforms

$f(t)$	$F(s) = \mathscr{L}\{f\}(s)$
1	$\frac{1}{s}, \quad s > 0$
e^{at}	$\frac{1}{s-a}, \quad s > a$
$t^n, \quad n = 1, 2, \ldots$	$\frac{n!}{s^{n+1}}, \quad s > 0$
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$e^{at}t^n, \quad n = 1, 2, \ldots$	$\frac{n!}{(s-a)^{n+1}}, \quad s > a$
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When Does the Laplace Transform of a Function Exist?

Piecewise Continuity

Definition 2. A function $f(t)$ is said to be **piecewise continuous on a finite interval** $[a, b]$ if $f(t)$ is continuous at every point in $[a, b]$, except possibly for a finite number of points at which $f(t)$ has a jump discontinuity.

A function $f(t)$ is said to be **piecewise continuous on** $[\mathbf{0}, \boldsymbol{\infty})$ if $f(t)$ is piecewise continuous on $[0, N]$ for all $N > 0$.

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When Does the Laplace Transform of a Function Exist?

Example 7 Show that $f(t) = \begin{cases} t, & 0 < t < 1, \\ 2, & 1 < t < 2, \\ (t-2)^2, & 2 \le t \le 3, \end{cases}$ is piecewise continuous on $[0, 3]$.

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When Does the Laplace Transform of a Function Exist?

Exponential Order α

Definition 3. A function $f(t)$ is said to be of **exponential order α** if there exist positive constants T and M such that

(4) $$|f(t)| \leq Me^{\alpha t}, \qquad \text{for all } t \geq T.$$

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When Does the Laplace Transform of a Function Exist?

Conditions for Existence of the Transform

Theorem 2. If $f(t)$ is piecewise continuous on $[0, \infty)$ and of exponential order α, then $\mathcal{L}\{f\}(s)$ exists for $s > \alpha$.